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Anne de Roton

ON THE MEAN SQUARE OF THE ERROR TERM FOR AN EXTENDED SELBERG'S CLASS

par

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Abstract. — We are concerned with an estimate and a mean square theorem for the summatory function of a class of Dirichlet series. This extension of Selberg's class is a class of Dirichlet series satisfying a functional equation involving multiple gamma factors and, contrary to the class studied by Chandrasekharan and Narasimhan, a conjugate, which allows twisted functions to belong to this class. If $F(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ is a Dirichlet series satisfying such a functional equation and $E(x)$ is the associated error term (see (1) and (3) respectively), then we prove O-estimate for $E(x)$ and $\int_0^x |E(y)|^2 dy$. These results are similar to those of Chandrasekharan and Narasimhan but are applicable in cases where theirs are not.

1. Notations

We write, as usual, $s = \sigma + i\tau$ and $\overline{f}(s) = \overline{f(\overline{s})}$. We recall here the definition of the extended Selberg class $S^{\#}$ studied by J. Kaczorowski and A. Perelli in a series of papers (see [**KP0**] for an introduction). The class $S^{\#}$ consists of the non identilcally vanishing functions $F(s)$ satisfying the following conditions.

1. For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$;
2. For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order ;
3. $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \epsilon \overline{\Phi}(1-s), \text{ where } \Phi(s) = Q^s \gamma(s) F(s), \tag{1}$$

with $\gamma(s) = \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$

$$\text{where } \lambda_j > 0,\ \mu_j \in \mathbb{C} \text{ such that } \Re\mu_j \geq 0,\ Q > 0 \text{ and } |\epsilon| = 1. \tag{2}$$

The Selberg class S is the set of functions $F \in S^{\#}$ satisfying the two more following axioms :

– (Ramanujan hypothesis) For every $\varepsilon > 0$, $a_n \ll n^{\varepsilon}$;
– (Euler product) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

where $b_n = 0$ unless n is a positive power of prime, and $b_n \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Notations 1. — *The degree d of $F \in S^{\#}$ is defined as $d = 2\sum_{j=1}^{r} \lambda j$, its polar order m_F is the least integer m satisfying 2. We also write*

$$\xi = 2\sum_{j=1}^{r}\left(\mu_j - \frac{1}{2}\right) \text{ and } \beta = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j}.$$

Remark 1. — *The functional equation* (1) *may be writen under the form*

$$F(1-s) = \epsilon Q^{2s-1} \frac{\overline{\gamma}(s)}{\gamma(1-s)} \overline{F}(s).$$

We define $S(x)$ as the sum of the residues of the function $\frac{F(s)x^s}{s}$ and the error term $E(x)$ as

$$E(x) = \sum_{n\leq x}{}' a_n - S(x). \tag{3}$$

In case $m_F \neq 0$, $E(x)$ is the error term for the summatory function of the coefficients of the Dirichlet series $F(x)$.

Remark 2. — *Since $s = 1$ is the only singularity of the function F, we can write $S(x)$ under the form :*

$$S(x) = F(0) + \operatorname{Res}\left(\frac{F(s)}{s}x^s, 1\right) = F(0) + xP_F(\log x),$$

where P_F is a polynomial function of degree $m_F - 1$. The last equality results from a simple argument, the reader may consult [**Lan**]*, vorbemerkung über $R(x)$, p.697, for more details.*

2. Introduction and statement of the results

In this paper we shall give an estimate and a mean square estimate for the error term $E(x)$ associated with a function F in the extended Selberg class $S^{\#}$.

Estimates for $E(x)$ and $\int_1^x |E(y)|^2 dy$ were first obtained by Voronoï in 1904 and Cramér in 1922 respectively for the special case of the divisor problem (see [**V**] and [**Cr**]). Since then, this has been generalised for larger classes of Dirichlet series.

As early as 1912, Landau gave an estimate of $E(x)$ for a class of Dirichlet series satisfying a functionnal equation of general type involving multiple gamma factors. The method of his proof is based on an application of the Perron formula and the residue theorem. A kind of Van der Corput method is also used to estimate the integrals.

In 1964, Chandrasekharan and Narasimhan gave an estimate for $E(x)$ and $\int_1^x |E(y)|^2 dy$ for functions of general type too (see [**CN1**], [**CN2**], [**CN3**]). Their method is based on the study of some hypergeometric functions. Since then, their results have been improved for functions of high degree by Redmond in [**R1**] and [**R2**].

Nethertheless, as far as we know, no estimate has been given in case of the extended Selberg class. The large classes already studied in the literature do not recover the class $S^{\#}$, since in the functionnal equations which have been considered, either the μ_j have to be reel, or no conjugate appear.

The results we obtain for a function of $S^{\#}$ are the same as the one obtained by Landau, Chandrasekharan and Narasimhan in case the μ_j are reel numbers. We shall prove the following theorems.

Theorem 1. — *Let $F \in S^{\#}$ be a function of degree $d \geq 2$. Then for $x \geq 1$, for all $\varepsilon > 0$ we have*

$$E(x) = O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right).$$

Theorem 2. — *Let $F \in S^{\#}$ such that for every $\varepsilon > 0$, $\sum_{n \leq x} |a_n|^2 = O_\varepsilon\left(x^{1+\varepsilon}\right)$. Then for $x \geq 1$, for all $\varepsilon > 0$ we have $E(x) = O\left(x^{1-1/d+\varepsilon}\right)$.*
If we assume furthermore that for all $\varepsilon > 0$, $a_n = O\left(n^\varepsilon\right)$, then we have for all $\varepsilon > 0$,

$$E(x) = O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right).$$

Theorem 3. — *Assume that $F \in S^{\#}$ such that for every $\varepsilon > 0$, $\sum_{n \leq x} |a_n|^2 = O_\varepsilon\left(x^{1+\varepsilon}\right)$. Then, for $x \geq 1$*

$$\begin{aligned} \int_1^x |E(y)|^2 dy &= O\left(x^{2-1/d}\right) && \text{if } 0 < d < 3 \\ &= O\left(x^{3-4/d+\varepsilon}\right) && \text{if } d \geq 3, \varepsilon > 0. \end{aligned}$$

The Selberg class was first introduced in [**S**] and have been studied since then in different papers by Conrey and Gosh (see [**CG1**], [**CG2**]), Murty (see [**M**]), Kaczorowski and Perelli (see [**KP0**], [**KP1**], [**KP2**]...). The structure of the extended Selberg class $S^{\#}$ we are considering here have been particularly studied by the two last authors. In [**KP1**] and [**KP2**], they study some incomplete Fox hypergeometric functions and establish a link between these functions and those of $S^{\#}$.

Many conjectures have been settled about S and $S^{\#}$. Especially $F \in S$ is supposed to verify the generalised Lindelöf hypothesis, i.e. :

Conjecture 1. —

$$\forall \varepsilon > 0,\ F\left(\frac{1}{2} + i\tau\right) \ll_\varepsilon \tau^\varepsilon.$$

This is wrong for a function in $S^{\#}$ (see [**CG2**]). This conjecture is strongly connected with the study of the error term $E(x)$ since we can prove that a function F in S satisfies the generalised Lindelöf hypothesis if and only if

$$\forall k \in \mathbb{N},\ \frac{F^k\left(\frac{1}{2}+i\tau\right)}{\frac{1}{2}+i\tau} \in L^2(\mathbb{R}),$$

and we prove in [**DR1**] that if $F \in S^{\#}$, then we have

$$\frac{F\left(\frac{1}{2}+i\tau\right)}{\frac{1}{2}+i\tau} \in L^2(\mathbb{R}) \Leftrightarrow E \in L^2([0,+\infty[, dx/x^2),$$

where E is the error term associated to F.

With our results, we prove that $E \in L^2([0,+\infty[, dx/x^2)$ for functions in $S^{\#}$ of degree less than 4. This result will be of the greater interest in the paper [**DR1**].

To prove theorem 1, we follow the method of Landau in [**Lan**]. Few arguments have to be modified. The methods of proof of theorems 2 and 3 both involve an identity relating a smoothed version of the error term to be estimated to a series of hypergeometric functions. We then study those hypergeometric functions and expand them asymptotically. To go back to the initial function, we use finite differences. This method was the one used by Chandrasekharan and Narasimhan in [**CN1**] and [**CN2**] except for the study of the hypergeometric function which was easier in their case. We have to connect our hypergeometric function to the Bessel functions to obtain some results similar to theirs.

We begin to give some technical lemmas we shall need further to prove our theorems. In section 3, we draw up a list of consequences of the complex Stirling formula and we especially give some estimates of a function of Selberg class in some vertical strips. In section 4, we give some technical lemmas. Section 5 will be devoted to the study of Bessel functions. In section 6, we connect a smoothed version of the error term with an hypergeometric funtion we study in section 7. In section 8 and 9, we prove the theorem 1 and 2 and theorem 3 respectively.

The author would like to thank the referee for his very fruitful comments, for pointing out several inaccuracies and for suggesting how to improve our results, especially in section 8. We also would like to thank Professor J. Kaczorowski for having suggested us to use Bessel functions during a very fruitful conversation in Cetraro.

3. Some consequences of the Stirling formula

We give here some technical estimates coming from the following Stirling formula. The reader will find a demonstration of this result in [**Bo**] (formule (19), §VII.2.3).

Proposition 3.1. — *There exists some constants $c_\nu = c_\nu(a)$, such that, for all $M \in \mathbb{N}$,*

$$\log\Gamma(s+a) = \left(s+a-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \sum_{\nu=1}^{M} c_\nu s^{-\nu} + O\left(\frac{1}{|s|^{M+1}}\right) \tag{4}$$

uniformly for $|\arg(s)| \le \pi - \varepsilon$ with a fixed $\varepsilon > 0$ and a in a compact of $\mathbb{C}$, $|s|$ tending to infinity.

3.1. Study of a function coming from the functionnal equation. —

Lemma 1. — *Let $\gamma(s)$ be as in (2). Then, uniformly for $s \in \mathbb{C}$ such that $\varepsilon \le |\arg s| \le \pi - \varepsilon$, for all $M \in \mathbb{N}$, we have when $|s|$ tends to infinity :*

$$\log\left(\frac{\overline{\gamma}(s)}{\gamma(1-s)}\right) = \frac{d}{2}s(\log s + \log(-s)) - s(\log\beta + d) + \frac{1}{2}\overline{\xi}\log s - \frac{1}{2}(\xi + d)\log(-s)$$
$$+\frac{1}{2}\log\beta + 2i\sum_{j=1}^{r}\Im\mu_j\log\lambda_j + \sum_{\nu=1}^{M} c_\nu s^{-\nu} + O\left(|s|^{-M-1}\right).$$

This follows easily from the Stirling formula.

Lemma 2. — *Let $\gamma(s)$ be as in (2). Then, uniformly for $\sigma_1 \le \sigma \le \sigma_2$, we have, as $|\tau|$ tends to infinity*

$$\frac{\overline{\gamma}(s)}{\gamma(1-s)} = c(\sigma,\tau)|\tau|^{d(\sigma-1/2)}e^{i(d\tau\log|\tau| - (\log\beta+d)\tau - \Im\xi\log|\tau|)}\left(1 + O\left(\frac{1}{|\tau|}\right)\right)$$

where $c(\sigma,\tau)$ is a complex number which only depends on σ and the sign of τ.

Notations 2. — *For $a > 0$ and $\nu \in \mathbb{R}$, we define*

$$f(s) = f_{a,\nu}(s) := \frac{1}{2}\left(\frac{2}{a}\right)^{s-\nu}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\nu - \frac{s}{2} + 1\right)}.$$

We shall establish a link between the function f and the multiple gamma factor γ.

Lemma 3. — *For $\alpha > 0$, $a > 0$, $\nu \in \mathbb{R}$, $\kappa \in \mathbb{C}$, there exists real numbers $K = K(\alpha, a, \kappa, \nu)$ and $c'_j = c'_j(\alpha, a, \kappa, \nu)$, $j \in \mathbb{N}$ such that for all $M \in \mathbb{N}$,*

$$\log(f(\alpha s + \kappa)) = \frac{\alpha}{2}s\left(\log s + \log(-s)\right) + \alpha(\ln\alpha - \ln a - 1)s + \frac{\kappa - 1}{2}\log s$$
$$+\left(\frac{\kappa-1}{2} - \nu\right)\log(-s) + K + \sum_{j=1}^{M} c'_j s^{-j} + O\left(\frac{1}{|s|^{M+1}}\right) \tag{5}$$

uniformly for s such that $\varepsilon < |\arg s| < \pi - \varepsilon$ with $\varepsilon > 0$, when $|s|$ tends to infinity.

We have in particular when $|\tau|$ tends to infinity

$$|f(s)| \asymp \left(|\tau|^{\sigma-1-\nu}\right), \tag{6}$$

uniformly for $\sigma \in [\sigma_1, \sigma_2]$.

Definition 1. — *For $\rho \in \mathbb{Z}$, $\rho \geq -1$, we define*

$$G_\rho(s) = \frac{\Gamma(1-s)\overline{\gamma}(s)}{\Gamma(2+\rho-s)\gamma(1-s)}.$$

By Stirling's formula, there exists a complex sequence $c_\nu^{(1)} = c_\nu^{(1)}(\lambda_1, \cdots, \lambda_r; \mu_1, \cdots, \mu_r, \rho)$, such that for all $M \in \mathbb{N}^*$,

$$\log G_\rho(s) = \frac{d}{2}s\big(\log s + \log(-s)\big) + s\left(\sum_{j=1}^r 2\lambda_j \log\lambda_j - d\right) + \frac{1}{2}\overline{\xi}\log s$$
$$-\left(\frac{1}{2}(\xi+d)+\rho+1\right)\log(-s) + \left(\sum_{j=1}^r(-\lambda_j - 2i\Im\mu_j)\log\lambda_j\right) + \sum_{\nu=1}^M c_\nu^{(1)} s^{-\nu} + O\left(\frac{1}{|s|^{M+1}}\right) \tag{7}$$

uniformly for $\varepsilon \leq |\arg(s)| \leq \pi - \varepsilon$ with $\varepsilon > 0$, if $|s|$ tends to infinity.

We have in particular

$$|G_\rho(s)| \asymp |\tau|^{d\left(\sigma-\frac{1}{2}\right)-\rho-1} \tag{8}$$

uniformly for $\sigma \in [\sigma_1, \sigma_2]$ and $|\tau|$ tending to infinity.

Notations 3. —

$$\alpha = d,\ \kappa = 1+\overline{\xi},\ \nu = 1 + \Re\xi + \frac{d}{2} + \rho,\ a = d\beta^{1/d}, \tag{9}$$

Comparing (7) to (5), we show that there exists some real numbers $e_k = e_k(\lambda_1, \cdots, \lambda_r; \mu_1, \cdots, \mu_r; \rho)$ such that, when $|\tau|$ tends to infinity, we have uniformly for $\sigma \in [\sigma_1, \sigma_2]$,

$$r_m(s) := G_\rho(s) - \sum_{k=0}^m F_k(s) = O\left(|f(\alpha s+\kappa)||s|^{-m-1}\right) = O\left(|\tau|^{d\left(\sigma-\frac{1}{2}\right)-\rho-m-2}\right)$$

with $F_k(s) = e_k f(\alpha s + \kappa)\frac{(-1)^k}{(\alpha s+\kappa-k)\cdots(\alpha s+\kappa-1)}$.

3.2. Estimates for the value of a quotient of Γ functions in half integers. —

Proposition 3.2. — *For $\nu \in \mathbb{R}$, $\Re z = N + \frac{1}{2}$ with $N \in \mathbb{N}$ and $N \geq |\nu| + 1/2$, we have :*

$$\frac{\Gamma(1-z)}{\Gamma(\nu+z)} \ll \frac{e^{2N}}{|z|^N|\nu+z|^{N+\nu}}.$$

Proof By the complement formula, we get :

$$\frac{\Gamma(1-z)}{\Gamma(\nu+z)} = \frac{\pi/\sin(\pi z)}{\Gamma(\nu+z)\Gamma(z)}.$$

For $z = N + 1/2 + it$, we have :

$$\left|\frac{\pi}{\sin(\pi z)}\right| = \left|\frac{2i\pi}{e^{i\pi(N+1/2)}e^{-\pi t} - e^{-i\pi(N+1/2)}e^{\pi t}}\right| = \frac{2\pi}{e^{-\pi|t|} + e^{\pi|t|}}.$$

On the other hand, by (4), we have uniformly for $\Re z \geq |\nu| + 1$,

$$\log|\Gamma(\nu+z)\Gamma(z)| = \left(\Re z - \frac{1}{2}\right)\log|z| + \left(\nu + \Re z - \frac{1}{2}\right)\log|\nu+z| - |\Im z|\pi$$

$$+\Im z\left(\arctan\frac{\Re z}{\Im z} + \arctan\frac{\Re z+\nu}{\Im z}\right) - 2\Re z - \nu + \log 2\pi + O\left(\frac{1}{|z|}\right)$$

$$\geq \left(\Re z - \frac{1}{2}\right)\log|z| + \left(\nu + \Re z - \frac{1}{2}\right)\log|\nu+z| - |\Im z|\pi - 2\Re z - \nu + \log 2\pi + O\left(\frac{1}{|z|}\right).$$

For $N \geq |\nu| + 1/2$ and $z = N + 1/2 + it$, we get

$$\log|\Gamma(\nu+z)\Gamma(z)| \geq N\log|z| + (\nu+N)\log|\nu+z| - |t|\pi - 2N - 1 - \nu + \log 2\pi + O\left(\frac{1}{|z|}\right).$$

So we have :

$$\frac{\Gamma(1-z)}{\Gamma(\nu+z)} \ll |z|^{-N}|\nu+z|^{-\nu-N}e^{2N}.$$

This ends the proof. □

3.3. Estimates of a function of the Selberg class in some vertical strips. —

Lemma 4. — *– Let $\varepsilon > 0$. Then we have $|F(s)| = O_\varepsilon(1)$ uniformly for $\sigma \geq 1+\varepsilon$ and $\tau \in \mathbb{R}$.*

– For $\sigma < 0$, we have $F(s) = O_\sigma\left((1+|\tau|)^{d(1/2-\sigma)}\right)$.

– For all $\varepsilon > 0$, if $-\varepsilon \leq \sigma \leq 1+\varepsilon$, we have $F(s) = O_{\sigma,\varepsilon}\left((1+|\tau|)^{\frac{d}{2}(1-\sigma+\varepsilon)}\right)$.

Proof The first estimation comes from the fact that $F(s)$ is an absolutely convergent Dirichlet series for $\sigma > 1$. The functional equation and lemma 2 provide the second one. To prove the last estimation, we use a Phagmén-Lindelöf argument. □

4. Some more lemmas

Definition 2. — *If $T > 0$, $\Lambda, c \in \mathbb{R}$ such that $c \leq \Lambda$, we define the path*

$$C(T,c,\Lambda) :=]c - i\infty; c - iT; \Lambda - iT; \Lambda + iT; c + iT; c + i\infty[.$$

In order to move paths of integration, we shall use the following lemma.

Lemma 5. — *Let f be a meromorphic function on $\mathbb{C}$ such that :*

1. *$T_0 := \sup\{|\Im\rho|,\ \rho$ pole of $f\} < +\infty$,*
2. *$\Lambda_0 := \sup\{\Re\rho,\ \rho$ pole of $f\} < +\infty$*

3. *there exists $a > 0$ and $b \in \mathbb{R}$ such that, for all $\sigma_1 < \sigma_2$, we have, uniformly in $\sigma \in [\sigma_1, \sigma_2]$, $|f(s)| \ll |\tau|^{a\sigma-b-1}$ when $|\tau|$ tends to infinity.*

If T, c and Λ satisfy :

$$T > T_0,\ c < \frac{b}{a} \text{ and } \Lambda > \Lambda_0, \tag{10}$$

then, for $x > 0$, the integral

$$I(x) := \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} f(s) x^{-s} ds$$

is convergent and its value does not depend on T, c and Λ satisfying (10). Furthermore, for all $\varepsilon > 0$, we have $I(x) = O_\varepsilon\left(x^{-b/a+\varepsilon}\right)$, when x tends to infinity.

Proof Under the assumptions of the lemma, the integrals are convergent on the path $C(T, c, \Lambda)$. To get the full conclusion, it is enough to use the residue theorem between the two paths $C(T, c, \Lambda)$ and $C(T', c', \Lambda')$ with T, c, Λ and T', c', Λ' satisfying (10) and observe that the horizontal integrals tend to zero. □

Lemma 6. — *Let $a, b \in \mathbb{R}^{+*}$ such that $b \geq 1$. Then we have :*

$$\int_0^{+\infty} \frac{d\tau}{(a^2 + \tau^2)^b} \leq \frac{\pi}{2} \frac{a^{1-2b}}{\sqrt{b}}.$$

Proof Since $a > 0$ and $b > \frac{1}{2}$, the integral is convergent. Furthermore, we have :

$$\int_0^{+\infty} \frac{d\tau}{(a^2 + \tau^2)^b} = a^{1-2b} \int_0^{+\infty} \frac{d\tau}{(1 + \tau^2)^b}.$$

By MAPLE, we get :

$$\int_0^{+\infty} \frac{d\tau}{(1 + \tau^2)^b} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(b - 1/2)}{\Gamma(b)}.$$

The logaritmic derivate of the function $\varphi(b) = \sqrt{b}\Gamma(b - 1/2)/\Gamma(b)$ is the function

$$\frac{\varphi'(b)}{\varphi(b)} = \frac{1}{2b} + \sum_{q \in \mathbb{N}} \left(\frac{1}{q + b} - \frac{1}{q + b - 1/2} \right) \leq \frac{1}{2b} + \int_0^{+\infty} \left(\frac{1}{t + b} - \frac{1}{t + b - 1/2} \right) dt \leq 0.$$

So we have for $b \geq 1$,

$$\int_0^{+\infty} \frac{d\tau}{(a^2 + \tau^2)^b} = a^{1-2b} \frac{\sqrt{\pi}}{2} \frac{\varphi(b)}{\sqrt{b}} \leq \frac{\sqrt{\pi}}{2} \frac{a^{1-2b}}{\sqrt{b}} \varphi(1) = \frac{\pi}{2} \frac{a^{1-2b}}{\sqrt{b}}. \qquad \square$$

5. Bessel functions

Our main reference for this section is the book of Watson [**W**]. We recall here some properties of the Bessel function and we give an integral representation of these functions which is not given in [**W**].

Notations 4. — *From now on, if c is a real number, we shall write $\int_{(c)} f(s)ds$ for $\int_{c-i\infty}^{c+i\infty} f(s)ds$.*

Let ν be a real number. We define the Bessel function J_ν as (formula (8) §3.1 of [**W**]) :

$$\forall x \in \mathbb{R}, \quad J_\nu(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

We recall the asymptotic expansion of $J_\nu(x)$ when x tends to infinity (§7.21 of [**W**]).

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu,2m)}{(2x)^{2m}} - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu,2m+1)}{(2x)^{2m+1}}\right)$$

with $(\nu,m) = \frac{\Gamma\left(\nu+m+\frac{1}{2}\right)}{m!\Gamma\left(\nu-m+\frac{1}{2}\right)}$.

From a classical Mellin transform result (formula (7) §6.5 of [**W**]), for $\nu \geq 0$ and $0 < c \leq \Re(\nu+1)$, we have

$$x^{-\nu} J_\nu(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\nu-1}\Gamma(s/2)}{\Gamma(\nu-s/2+1)} x^{-s} ds.$$

Remark 3. — *We recover here the function $f_{a,\nu}$ defined and studied in the section 3.1.*

We will extend this result to the case $\nu \leq 0$.

Lemma 7. — *Let $a > 0$ and $\nu \geq 0$. If $f_{a,\nu}$ is defined as in section 3.1, then we have*

$$\lim_{N\to+\infty} \int_{(-2N+1)} f_{a,\nu}(s) x^{-s} ds = 0,$$

uniformly for x in a compact of $\mathbb{R}^{+*}$.

Proof We define $K_N(x) := \int_{(-2N+1)} f_{a,\nu}(s)x^{-s}ds$. For $N \geq |\nu|+1$, with $z = 1 - \frac{s}{2}$ and proposition 3.2, we have :

$$K_N(x) = \frac{1}{2} \int_{(N+1/2)} \left(\frac{2}{a}\right)^{2-2z-\nu} \frac{\Gamma(1-z)}{\Gamma(\nu+z)} x^{2z-2} dz = O\left(\frac{1}{x}\left(\frac{eax}{2}\right)^{2N} \int_{(N+1/2)} \frac{|dz|}{|z|^N|\nu+z|^{N+\nu}}\right).$$

Furthermore, if $u, v \geq 1$, we have $\frac{1}{uv} \leq \frac{1}{u} + \frac{1}{v}$, so we get :

$$K_N(x) = O\left(\frac{1}{x}\left(\frac{eax}{2}\right)^{2N}\left(\int_{(N+1/2)} \frac{|dz|}{|z|^N} + \int_{(N+1/2)} \frac{|dz|}{|\nu+z|^{N+\nu}}\right)\right).$$

With lemma 6, we have
$$\int_{(N+1/2)} \frac{|dz|}{|z|^N} = \int_{-\infty}^{+\infty} \frac{dt}{\left((N+1/2)^2+t^2\right)^{N/2}} \ll \frac{(N+1/2)^{1-N}}{\sqrt{N}}$$
and
$$\int_{(N+1/2)} \frac{|dz|}{|\nu+z|^{N+\nu}} = \int_{-\infty}^{+\infty} \frac{dt}{\left((N+\nu+1/2)^2+t^2\right)^{(N+\nu)/2}} \ll \frac{(N+\nu+1/2)^{1-N-\nu}}{\sqrt{N+\nu}}.$$
If $N \geq 2|\nu|+1$, we get :
$$K_N(x) = O\left(\frac{1}{x}\sqrt{N}\left(\frac{eax}{2}\right)^{2N}\left(N^{-N}+(N/2)^{-N/2}\right)\right),$$
and the integral $K_N(x) := \int_{(-2N+1)} f(s)x^{-s}ds$ tends clearly to 0 when N tends to infinity uniformly for x in a compact of $\mathbb{R}^{+*}$. □

Definition 3. — *Let $a>0$ and $\nu \in \mathbb{R}$. We define the function $h_0 = h_0^{\nu,a}$ by :*
$$\forall x>0, \quad h_0(x) := x^{-\nu}J_\nu(ax).$$

Proposition 5.1. — *If $\nu \in \mathbb{R}$ and if*
$$T \geq 1, \; c<\nu \text{ and } \Lambda > 0, \; \Lambda \geq c, \tag{11}$$
then the integral $\int_{C(T,c,\Lambda)} f(s)x^{-s}ds$ is convergent and its value does not depend on T, c and Λ satisfying (11). Furthermore, we have :
$$h_0(x) = \frac{1}{2i\pi}\int_{C(T,c,\Lambda)} f_{a,\nu}(s)x^{-s}ds.$$

Proof In this proof, we will write f for $f_{a,\nu}$.
Since the poles of the function $f_{a,\nu}(\alpha s+\kappa)$ are the points $\frac{-2n-\kappa}{\alpha}$, $n \in \mathbb{N}$, the first part of the proposition follows from (6) and lemma 5.

We define
$$g(x) := \frac{1}{2i\pi}\int_{C(T,c,\Lambda)} f(s)x^{-s}ds.$$
If N is a positive integer such that $N > \frac{1}{2}(1-\nu)$, then
$$g(x) = \frac{1}{2i\pi}\int_{C(T,-2N+1,\Lambda)} f(s)x^{-s}ds.$$
We apply the residue theorem to the function $f(s)x^{-s}$ on the rectangle $R := [-2N+1-iT, \Lambda-iT, \Lambda+iT, -2N+1+iT]$. We get :
$$g(x) = \frac{1}{2i\pi}\int_{(-2N+1)} f(s)x^{-s}ds + \sum_{k=0}^{N-1} p_k x^{2k}, \tag{12}$$
where $p_k = \mathrm{Res}(f(s)x^{-s}, -2k) = \left(\frac{2}{a}\right)^{-2k-\nu}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}$.
From lemma 7, the integral $K_N(x) := \int_{(-2N+1)} f(s)x^{-s}ds$ tends to 0 when N tends

to infinity uniformly for x in a compact of $\mathbb{R}^{+*}$. If N tend to infinity in (12), we get $g(x) = \sum_{m\geq 0} p_m x^{2m} = x^{-\nu} J_\nu(ax)$. $\square$

We shall use some primitive functions of h_0 in the case $\nu > 0$. To define them, we use some properties of cosine and sine functions.

Definition 4. — *Let a and ν be some real such that $a > 0$. For $t \in \mathbb{R}$, we define*

$$C(t) = \cos\left(at - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad S(t) = \sin\left(at - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and for $x > 0$ and $R > -\frac{1}{2}$,

$$\mathcal{C}_R(x) = -\int_x^{+\infty} \sqrt{\frac{2}{\pi}} C(t) \frac{dt}{(at)^{R+\frac{1}{2}}}; \quad \mathcal{S}_R(x) = -\int_x^{+\infty} \sqrt{\frac{2}{\pi}} S(t) \frac{dt}{(at)^{R+\frac{1}{2}}}.$$

Using induction and partial integration, we establish the following asymptotic expansions. For all $n \in \mathbb{N}$ and all $R > -\frac{1}{2}$, if for $k \in [0, n]$, $\gamma_k^{(R)} := \frac{\Gamma\left(R+k+\frac{1}{2}\right)}{\Gamma\left(R+\frac{1}{2}\right)}$, then

$$\begin{aligned}\mathcal{C}_R(x) = \frac{1}{a}\sqrt{\frac{2}{\pi}} \Bigg(S(x) \sum_{k=0}^{n} (-1)^k \gamma_{2k}^{(R)} (ax)^{-R-2k-\frac{1}{2}} \\ + C(x) \sum_{k=0}^{n-1} (-1)^{k+1} \gamma_{2k+1}^{(R)} (ax)^{-R-2k-\frac{3}{2}} \Bigg) + O\left(x^{-R-2n-\frac{3}{2}}\right), \quad (13)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_R(x) = -\frac{1}{a}\sqrt{\frac{2}{\pi}} \Bigg(C(x) \sum_{k=0}^{n} (-1)^k \gamma_{2k}^{(R)} (ax)^{-R-2k-\frac{1}{2}} \\ + S(x) \sum_{k=0}^{n-1} (-1)^{k} \gamma_{2k+1}^{(R)} (ax)^{-R-2k-\frac{3}{2}} \Bigg) + O\left(x^{-R-2n-\frac{3}{2}}\right), \quad (14)\end{aligned}$$

when x tends to infinity. The implicit constants only depend on R and n.

We can now deduce from (13) and (14) the following property.

Proposition-Definition 1. — *For $\nu > 0$, we define by induction some primitive of h_0 of order n*

$$h_{n+1}(x) = -\int_x^{+\infty} h_n(t)dt.$$

These integrals are convergent and for all $k \in \mathbb{N}$ and there exists two complex sequences $(c_l^{(k)})_l$ and $(s_l^{(k)})_l$ such that for all $M \in \mathbb{N}$,

$$h_k(x) = \sum_{l=0}^{M} c_l^{(k)} C(x) x^{-l-\nu-\frac{1}{2}} + \sum_{l=0}^{M} s_l^{(k)} S(x) x^{-l-\nu-\frac{1}{2}} + O\left(x^{-\nu-M-\frac{3}{2}}\right)$$

when x tends to infinity.

Proposition 5.2. — *Let $k \in \mathbb{N}$ and $\nu > 0$. So, if $0 < c < \nu$ and $x > 0$,*

$$h_k(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} (-1)^k \frac{\Gamma(s)}{\Gamma(s+k)} f(s+k) x^{-s} ds.$$

Proof We begin to prove by induction on k that for all $s \in \mathbb{C}$ such that $0 < \sigma < \nu + 1/2$,

$$\mathcal{M}h_k(s) = (-1)^k \frac{\Gamma(s)}{\Gamma(s+k)} f(s+k).$$

Then we use inverse Mellin transform. □

6. Expansion in a sum of hypergeometric functions

In order to estimate the error term $E(x)$, we define a smooth version of it. To go back to the initial function, we shall use the finite differences.

Definition 5. — *For $\rho \in \mathbb{N}$, we define*

$$A^\rho(x) = \frac{1}{\rho!} \sum_{n \leq x}{}' a_n (x-n)^\rho,$$

the dash indicating that the last term has to be multiplied by $1/2$, if $\rho = 0$ and $x = n$. We also define

$$S_\rho(x) = \frac{1}{2i\pi} \int_C \frac{F(s)}{s(s+1)\cdots(s+\rho)} x^{s+\rho} ds,$$

where C is a curve enclosing all the singularities of the integrand, and

$$E^\rho(x) = A^\rho(x) - S^\rho(x).$$

Remark 4. — *$S_\rho(x)$ is the sum of the residues of the integrand at its poles $-\rho, -\rho + 1, \cdots, -1, 0$ and 1, so we have*

$$S_\rho(x) = x^\rho \mathrm{Res}\left(\frac{F(s)x^s}{s(s+1)\cdots(s+\rho)}, 1\right) + \frac{1}{\rho!} \sum_{k=0}^{\rho} (-1)^k F(-k) \binom{\rho}{k} x^{\rho-k}.$$

The first step consists in finding an expansion of $E^\rho(x)$ as a sum of special functions.

Proposition 6.1. — *Let $\rho \in \mathbb{N}^*$, $\rho > d/2$. For all $x > 0$,*

$$E^\rho(x) = \epsilon Q^{2\rho+1} \sum_{n \geq 1} \frac{\overline{a_n}}{n^{1+\rho}} I_\rho\left(\frac{nx}{Q^2}\right)$$

where for $\rho \in \mathbb{Z}$, $\rho \geq -1$, the function I_ρ is defined as

$$I_\rho(x) = \frac{1}{2i\pi} \int_{C_1} \frac{\overline{\gamma}(s)}{\gamma(1-s)} \frac{x^{1+\rho-s}\Gamma(1-s)}{\Gamma(\rho+2-s)} ds$$

and $C_1 = C(T, c, \Lambda)$ with

$$T > T_0 := \max\left(\frac{|\Im\xi|}{d}, \max_{j=1}^{r} \frac{|\Im\mu_j|}{\lambda_j}\right), \ c < \frac{\rho}{d} + \frac{1}{2} \text{ and } \Lambda > \rho + 1. \tag{15}$$

Proof If $c > 1$ and $\rho \geq 1$, then by a standard formula, we have

$$A^\rho(x) = \frac{1}{2i\pi} \int_{\Re s = c} F(s) \frac{x^{s+\rho}}{s(s+1)\cdots(s+\rho)} ds,$$

so that, if $g(s) = \frac{F(s)}{s(s+1)\cdots(s+\rho)}$, we have

$$E^\rho(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} g(s) x^{s+\rho} ds - \frac{1}{2i\pi} \int_C g(s) x^{s+\rho} ds.$$

In view of lemma 4, we know that if $\sigma < 0$ and $|\tau| \geq 1$, then $g(s) = O\left((1+|\tau|)^{d(1/2-\sigma)-\rho-1}\right)$, so that, if $\frac{1}{2} - \frac{\rho}{d} < c' < 0$ (such a c' exists if $\rho > \frac{d}{2}$), $\Lambda' < -\rho$, $T' > 0$ and $C_2 :=]c' - i\infty; c' - iT'; \Lambda' - iT'; \Lambda' + iT'; c' + iT'; c' + i\infty[$, then the integral

$$\frac{1}{2i\pi} \int_{C_2} \frac{F(s)}{s(s+1)\cdots(s+\rho)} x^{s+\rho} ds$$

is convergent and by using the residue theorem between the line $\sigma = c$ and C_2, we get

$$E^\rho(x) = \frac{1}{2i\pi} \int_{C_2} \frac{F(s)}{s(s+1)\cdots(s+\rho)} x^{s+\rho} ds.$$

We change the variable s to $1-z$ and apply the functional equation, to obtain

$$E^\rho(x) = \frac{1}{2i\pi} \int_{C_1} \frac{\epsilon Q^{2z-1} \overline{F}(z)}{(1-z)(-z)\cdots(1-z+\rho)} x^{1-z+\rho} \frac{\overline{\gamma}(z)}{\gamma(1-z)} dz$$

where $C_1 = C(T, c, \Lambda)$ with

$$T > 0,\ c < \frac{\rho}{d} + \frac{1}{2} \text{ and } \Lambda > \rho + 1.$$

By lemma 2 and lemma 5, we get for $T > T_0 = \max\left(\frac{|\Im \xi|}{d}, \max_{j=1}^r \frac{|\Im \mu_j|}{\lambda_j}\right)$, $1 < c < \frac{\rho}{d} + \frac{1}{2}$ and $\Lambda > \rho + 1$:

$$E^\rho(x) = \epsilon Q^{2\rho+1} \sum_{n=1}^{+\infty} \frac{\overline{a_n}}{n^{1+\rho}} \frac{1}{2i\pi} \int_{C_1} \frac{\overline{\gamma}(z)}{\gamma(1-z)} \frac{\left(\frac{nx}{Q^2}\right)^{1+\rho-z}}{(1-z)(2-z)\cdots(1+\rho-z)} dz \qquad \square$$

7. Study of the hypergeometric function I_ρ

We now study the function I_ρ. Kaczorowski and Perelli have studied a very similat function in [**KP1**] in order to describe functions of gegre $d = 1$ in $S^\#$. Our work is strongly influenced by theirs. This function is a particular case of the Fox hypergeometric functions. These last functions have been studied in [**Br**]. We will rather study the particular function I_ρ instead of extracting the informations we need from the very long and dense article [**Br**]. In order to establish the asymptotic expansion of the function I_ρ, we show that this function is closely related to Bessel functions.

We shall use the notations defined in section 3.1.

The function I_ρ is closely related to the Mellin inverse transform of G_ρ. We are going to approach G_ρ with functions whose inverse Mellin transforms are known.

We recall here that we have proved that

$$G_\rho(s) = \sum_{k=0}^{m} F_k(s) + r_m(s),$$

with $F_k(s) = \dfrac{(-1)^k e_k f(\alpha s+\kappa)}{(\alpha s+\kappa-k)\cdots(\alpha s+\kappa-1)}$ and $r_m(s) = O\left(|\tau|^{d\left(\sigma-\frac{1}{2}\right)-\rho-m-2}\right)$.

If $T > T_0$, $\Lambda > \max\left(\rho+1, \frac{m-\Re\kappa}{d}\right)$, $c < \frac{1}{2} + \frac{\rho}{d}$, if c is not a pole of r_k and if $C_1 = C(T, c, \Lambda)$ is the path defined in (2), then the integrals

$$H_k(x) := \frac{1}{2i\pi} \int_{C_1} F_k(s) x^{1+\rho-s} ds \text{ and } R_k(x) := \int_{C_1} r_k(s) x^{1+\rho-s} ds$$

are convergent. With these notations, we have for $x > 0$ and $m \in \mathbb{N}$

$$I_\rho(x) = \sum_{k=0}^{m} H_k(x) + R_m(x). \tag{16}$$

To estimate R_m, we move the path of integration to the right and pay attention to the fact that the integral remains convergent. By proposition 5, we get for $m \in \mathbb{N}$ and $\varepsilon > 0$

$$R_m(x) = O_\varepsilon\left(x^{\frac{1}{2}+\rho-\frac{\rho+m+1}{d}+\varepsilon}\right) \tag{17}$$

when x tends to infinity.

Proposition 7.1. — *If $T \geq 1$, $\Lambda > \max(0, c)$ and $c < \nu$, we have*

$$H_k(x) = \frac{e_k}{d2i\pi} x^{1+\rho+(\kappa-k)/d} \int_{C(T,c,\Lambda)} \frac{(-1)^k f(z+k)}{z(z+1)\cdots(z+k-1)} (x^{1/d})^{-z} dz.$$

Proof By (6), when $|\tau|$ tends to infinity

$$F_k(s) \ll |\tau|^{d\left(\sigma-\frac{1}{2}\right)-\rho-k-1},$$

so if $T > T_0 := \max\left(\frac{|\Im\xi|}{d}, \max_{j=1}^{r} \frac{|\Im\mu_j|}{\lambda_j}\right)$, $c < \frac{1}{d}(\rho+k)+\frac{1}{2}$ and $\Lambda > \max\left(\rho+1, \frac{k-\Re\kappa}{d}\right)$, by proposition 5, we have

$$\begin{aligned} H_k(x) &= \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} F_k(s) x^{1+\rho-s} ds \\ &= \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} e_k f(\alpha s+\kappa) \frac{(-1)^k x^{1+\rho-s}}{(\alpha s+\kappa-k)\cdots(\alpha s+\kappa-1)} ds. \end{aligned}$$

We change the variable s to $z = \alpha s + \kappa - k$

$$H_k(x) = \frac{e_k/d}{2i\pi} x^{1+\rho+(\kappa-k)/d} \int_{C(T',c',\Lambda')} \frac{(-1)^k f(z+k)}{z(z+1)\cdots(z+k-1)} (x^{1/d})^{-z} dz$$

for any real T', c' and Λ' satisfying $T' \geq 1$, $\Lambda' > \max(0, c')$ and $c' < \nu$. $\square$

We now establish a link between H_k and h_k in case $\nu > 0$, i.e. in case $\rho > -\Re\kappa - \frac{d}{2}$.

Proposition 7.2. — *If $\nu > 0$, for $x > 0$ we have*

$$H_k(x) = \frac{e_k}{d} x^{1+\rho+\frac{\kappa-k}{d}} h_k(x^{1/d}).$$

Proof By proposition 7.1, we have

$$H_k(x) = \frac{e_k/d}{2i\pi} x^{1+\rho+(\kappa-k)/d} \int_{C(T,c,\Lambda)} \frac{(-1)^k f(z+k)}{z(z+1)\cdots(z+k-1)} (x^{1/d})^{-z} dz.$$

By proposition 5, since the real part of the poles of the integrand is less than 0, since we have

$$\left| \frac{(-1)^k f(z+k)}{z(z+1)\cdots(z+k-1)} \right| \asymp |\tau|^{\sigma-\nu-1},$$

we can chose $0 < c = \Lambda < \nu$ because $\nu > 0$ and then we have

$$\begin{aligned} H_k(x) &= \frac{e_k/d}{2i\pi} x^{1+\rho+(\kappa-k)/d} \int_{c-i\infty}^{c+i\infty} \frac{(-1)^k f(z+k)}{z(z+1)\cdots(z+k-1)} (x^{1/d})^{-z} dz \\ &= \delta e_k x^{1+\rho+\delta(\kappa-k)} h_k(x^{1/d}). \end{aligned}$$

□

For $k = 0$ and $\nu \in \mathbb{R}$, we also have

$$H_0(x) = \frac{e_0}{d} x^{1+\rho+\frac{\kappa}{d}} h_0(x^{1/d}). \tag{18}$$

We shall now deduce from the link between I_ρ and h_k an asymptotic expansion of I_ρ in case $\nu > 0$ and an asymptotic equivalent of I_0 in case $\nu \leq 0$.

Notations 5. — *We write $\delta = 1/d$, $\omega = 1+\rho+(\kappa-\nu-1/2)\delta = (\rho+1/2)(1-\delta)-i\delta\Im\xi$.*

Theorem 4. — *Assume $\rho > -1 - \Re\xi - \frac{d}{2}$ (ie $\nu > 0$). There exists two complex sequences $(\delta_n)_n$ and $(\delta'_n)_n$ such that for all $m \in \mathbb{N}$ and all $\varepsilon > 0$*

$$I_\rho(x) = e^{iax^\delta} \sum_{n=0}^{m} \delta_n x^{\omega-n\delta} + e^{-iax^\delta} \sum_{n=0}^{m} \delta'_n x^{\omega-n\delta} + O\left(x^{\Re\omega-(m+1/2)\delta+\varepsilon}\right)$$

when x tends to infinity.

Proof By proposition 7.2, (17) and (16), we have

$$\begin{aligned} I_\rho(x) &= \sum_{k=0}^{m} \delta e_k x^{1+\rho+(\kappa-k)\delta} h_k(x^\delta) + O\left(x^{\frac{1}{2}+\rho-(\rho+m+1)\delta+\varepsilon}\right) \\ &= \sum_{k=0}^{m} \delta e_k x^{1+\rho+(\kappa-k)\delta} h_k(x^\delta) + O\left(x^{\Re\omega-\delta(m+1/2)+\varepsilon}\right) \end{aligned}$$

Using the asymptotic expansion of functions h_k (see proposition-definition 1) we show that there exists two complex sequences $(\delta_n)_n$ and $(\delta'_n)_n$ such that for all $m \in \mathbb{N}$

$$\begin{aligned} I_\rho(x) = C\left(x^\delta\right) \sum_{n=0}^{m} \delta_n x^{1+\rho+\left(\kappa-\nu-n-\frac{1}{2}\right)\delta} + S\left(x^\delta\right) \sum_{n=0}^{m} \delta'_n x^{1+\rho+\left(\kappa-\nu-n-\frac{1}{2}\right)\delta} \\ +O\left(x^{\frac{1}{2}+\rho-(\rho+m+1)\delta+\varepsilon}\right) \end{aligned}$$

so there exists two complex sequences $(\eta_n)_n$ and $(\eta'_n)_n$ such that for all $m \in \mathbb{N}$

$$I_\rho(x) = e^{iax^\delta} \sum_{n=0}^{m} \delta_n x^{\omega - n\delta} + e^{-iax^\delta} \sum_{n=0}^{m} \delta'_n x^{\omega - n\delta} + O\left(x^{\Re\omega - (m+1/2)\delta + \varepsilon}\right) \qquad \square$$

Proposition 7.3. — *Let ν be a real number. We have when x tends to infinity*

$$I_0(x) = O\left(x^{\frac{1}{2}(1-\delta)}\right).$$

Proof By (16), (17) and (18), we have for all $\varepsilon > 0$:

$$I_0(x) = \delta e_0 x^{1+\delta\kappa} h_0(x^\delta) + O_\varepsilon\left(x^{\frac{1}{2}-\delta+\varepsilon}\right).$$

But $h_0(x) = O\left(x^{-\nu-\frac{1}{2}}\right) = O\left(x^{-\Re\kappa - \frac{d}{2} - \frac{1}{2}}\right)$, so $I_0(x) = O\left(x^{\frac{1}{2}(1-\delta)}\right) + O_\varepsilon\left(x^{\frac{1}{2}-\delta+\varepsilon}\right)$. We chose $0 < \varepsilon < \frac{\delta}{2}$ to get the final result. $\square$

Proposition 7.4. — *If $\rho \in \mathbb{N}$, the function I_ρ has a ρ^{th} derivative and for all $k \in \mathbb{N}$, $k \leq \rho$, we have*

$$I_\rho^{(k)}(y) = I_{\rho-k}(y).$$

Proof Assume that $\rho \in \mathbb{N}$ and T, c and Λ satisfy (15), then we have :

$$I_\rho(x) = \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} G_\rho(s) x^{1+\rho-s} ds.$$

We have

$$\frac{d}{dx}\left(G_\rho(s) x^{1+\rho-s}\right) = (1+\rho-s) G_\rho(s) x^{\rho-s}$$

and the integral

$$\int_{C(T,c,\Lambda)} (1+\rho-s) G_\rho(s) x^{\rho-s} ds$$

is convergent uniformly for x in a compact of $\mathbb{R}^+$. So we have

$$I'_\rho(x) = \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} (1+\rho-s) G_\rho(s) x^{\rho-s} ds.$$

But $(1+\rho-s)G_\rho(s) = G_{\rho-1}(s)$ so

$$I'_\rho(x) = \frac{1}{2i\pi} \int_{C(T,c,\Lambda)} G_{\rho-1}(s) x^{\rho-s} ds.$$

Since T, c and Λ satisfy conditions (15) with $\rho - 1$ instead of ρ, we have

$$I'_\rho(x) = I_{\rho-1}(x).$$

An iteration gives the conclusion. $\square$

8. Estimation of the error term

8.1. Estimation of the error term with Landau's method. — We shall prove in this section theorem 1.
Let $F \in S^{\#}$ be a function of degree $d \geq 2$.
Using arguments similar to the one used in the proof of the Perron formula (see [**T**], section I.2,theorem 1), we prove that for all $\varepsilon > 0$, $x \geq 1$ and $T \geq 1$, we have

$$\sum_{n\leq x}{}' a_n = \frac{1}{2i\pi}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{x^s F(s)}{s} ds + O\left(x^{1+\varepsilon} \sum_{\substack{n=1\\ n\neq x}}^{+\infty} \frac{|a_n|}{n^{1+\varepsilon}(T|\log(x/n)|)} \right) + O\left(\frac{|a_x|}{T}\right)$$

$$= \frac{1}{2i\pi}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{x^s F(s)}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(\frac{|a_x|}{T}\right),$$

where $a_x = 0$ if $x \notin \mathbb{N}$.

Applying the residue theorem, we get

$$\frac{1}{2i\pi}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{x^s F(s)}{s} ds = S(x) + \frac{1}{2i\pi}\int_{1+\varepsilon-iT}^{-\varepsilon-iT} \frac{x^s F(s)}{s} ds$$
$$+ \frac{1}{2i\pi}\int_{-\varepsilon-iT}^{-\varepsilon+iT} \frac{x^s F(s)}{s} ds + \frac{1}{2i\pi}\int_{-\varepsilon+iT}^{1+\varepsilon+iT} \frac{x^s F(s)}{s} ds.$$

Let us estimate the horizontal integrals. If $s = \sigma + iT$ with $T \geq 1$ and $-\varepsilon \leq \sigma \leq 1+\varepsilon$, by lemma 4, we have

$$\frac{x^s}{s} F(s) = O\left(T^{\frac{d}{2}(1+\varepsilon)-1} \left(xT^{-d/2}\right)^{\sigma} \right),$$

and therefore,

$$\int_{-\varepsilon\pm iT}^{1+\varepsilon\pm iT} \frac{x^s F(s)}{s} ds = O\left(T^{\frac{d}{2}(1+\varepsilon)-1} \left(\frac{x^{1+\varepsilon}}{T^{d(1+\varepsilon)/2}} + \frac{x^{-\varepsilon}}{T^{-d\varepsilon/2}} \right) \right)$$
$$= O\left(\frac{1}{T}x^{1+\varepsilon}\right) + O\left(x^{-\varepsilon}T^{d(1+2\varepsilon)/2-1}\right).$$

Now, let us estimate the vertical integral. Using the functionnal equation and lemma 2, we have

$$F(s) = \epsilon Q^{2s-1} \frac{\overline{\gamma}(1-s)}{\gamma(s)} \overline{F}(1-s)$$

$$= \epsilon Q^{2s-1} c(\sigma,\tau) |\tau|^{d(1/2-\sigma)} e^{i(-d\tau\log|\tau| + (\log\beta+d)\tau - \Im\xi\log|\tau|)} \overline{F}(1-s) \left(1 + O\left(\frac{1}{|\tau|}\right)\right).$$

So, with $s = -\varepsilon + i\tau$, we have

$$F(s)\frac{x^s}{s} = Ax^{-\varepsilon}|\tau|^{d(1/2+\varepsilon)-1} \sum_{n\geq 1} \frac{\overline{a_n}}{n^{1+\varepsilon}} e^{i\left(-(d\tau+\Im\xi)\log|\tau| + (\log\frac{\beta Q^2 x}{n}+d)\tau\right)} \left(1 + O\left(\frac{1}{|\tau|}\right)\right),$$

where A is a constant depending on ε and the sign of τ. We have

$$\int_{-\varepsilon-iT}^{-\varepsilon+iT} F(s)\frac{x^s}{s} = A_1 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1/d}^{T}\tau^{d(1/2+\varepsilon)-1}e^{i\left(-(d\tau+\Im\xi)\log\tau+(\log\frac{\beta Q^2 x}{n}+d)\tau\right)}d\tau$$

$$+A_2 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1/d}^{T}\tau^{d(1/2+\varepsilon)-1}e^{i\left((d\tau-\Im\xi)\log\tau-(\log\frac{\beta Q^2 x}{n}+d)\tau\right)}d\tau$$

$$+O\left(x^{-\varepsilon}\sum_{n\geq 1}\int_{1/d}^{T}\tau^{d(1/2+\varepsilon)-2}\right)+O(x^{-\varepsilon}).$$

With $u=d\tau$, we have

$$\int_{-\varepsilon-iT}^{-\varepsilon+iT} F(s)\frac{x^s}{s} = B_1 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1}^{dT}u^{d(1/2+\varepsilon)-1}e^{i\left(-(u+\Im\xi)\log u+(\log d+\log\frac{\beta Q^2 x}{n}/d+1)u\right)}du$$

$$+B_2 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1}^{dT}u^{d(1/2+\varepsilon)-1}e^{i\left((u-\Im\xi)\log u-(\log\frac{\beta Q^2 x}{n}/d+1+\log d)u\right)}du$$

$$+O\left(x^{-\varepsilon}T^{d(1/2+\varepsilon)-1}\right)+O(x^{-\varepsilon}).$$

$$= B_1 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1}^{dT}u^{d(1/2+\varepsilon)-1}e^{i(\varphi(u)-\Im\xi\log u)}du$$

$$+B_2 x^{-\varepsilon}\sum_{n\geq 1}\frac{\overline{a_n}}{n^{1+\varepsilon}}\int_{1}^{dT}u^{d(1/2+\varepsilon)-1}e^{i(-\varphi(u)-\Im\xi\log u)}du$$

$$+O\left(x^{-\varepsilon}T^{d(1/2+\varepsilon)-1}\right)+O(x^{-\varepsilon})$$

with $\varphi(u)=-u\log u+(\log\frac{\beta Q^2 x}{n}/d+1+\log d)u$.

According to Hilfsatz 10 of [**Lan**], we have for $U\geq 1$, $\delta\geq 0$ and $w\in\mathbb{R}$, the following equality

$$\left|\int_1^U u^{\delta}e^{\pm iu(\log u-w)}du\right|<23U^{\frac{1}{2}+\delta}.$$

Applying this lemma with $\delta=d(1/2+\varepsilon)-1$ and $w=\log\frac{\beta Q^2 x}{n}/d+1+\log d$, we get for $U\geq 1$ and $d\geq 2$

$$\int_1^U u^{d(1/2+\varepsilon)-1}e^{\pm i\varphi(u)}du<23U^{d(1/2+\varepsilon)-1/2}.$$

Therefore, we have

$$\int_1^{dT}u^{d(1/2+\varepsilon)-1}e^{i(\pm\varphi(u)-\Im\xi\log u)}du=O\left(T^{d(1/2+\varepsilon)-1/2}\right)$$

and so

$$\int_{-\varepsilon-iT}^{-\varepsilon+iT} F(s)\frac{x^s}{s}=O\left(x^{-\varepsilon}T^{d(1/2+\varepsilon)-1/2}\right).$$

Finaly, we have proved that if $d \geq 2$, we have

$$\sum_{n\leq x}{}' a_n = S(x) + O\left(\frac{1}{T}x^{1+\varepsilon}\right) + O\left(x^{-\varepsilon}T^{(d-1)/2+\varepsilon}\right) + O\left(\frac{|a_x|}{T}\right)$$

so with $T = x^{\frac{d+1}{2}}$, since $a_n = o(n^{1+\varepsilon})$, we have

$$\sum_{n\leq x}{}' a_n = S(x) + O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right).$$

Remark 5. — *With a similar method combined with the Van der Corput method used instead of Hilfsatz 10, we should prove that for $1 \leq d < 2$, we have the estimation $E(x) = O\left(x^{\frac{d}{4}+\varepsilon}\right)$.*
We did not manage to prove the estimation $E(x) = O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right)$ in the case $d < 2$. We shall prove it in the next subsection with the additionnal Ramanujan hypothesis. This hypothesis must be redundant but we did not manage to avoid it.
Anyway, we can notice that the degree of the functions of $S^{\#}$ is conjecturally a positive integer (see [**KP0**]*). Kaczorowski and Perelli have already proved that there are no function in $S^{\#}$ of degree $1 < d < 5/3$ (see* [**KP2**]*). Moreover, we know that the functions of $S^{\#}$ of degree $d \leq 1$ satisfy the Ramanujan hypothesis and, since Kaczorowski and Perelli have explicitly describded these functions, it is easy to prove that their error term satisfies $E(x) = O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right)$ (see* [**KP1**]*).*

8.2. Method used by Chandrasekharan and Narasimhan. — We shall use finite differences to establish a link between $E(x)$ and its smooth version $E^\rho(x)$.

If ρ is an integer, $\lambda > 0$ and $0 < \rho\lambda < x$, the ρ^{th} finite difference of the real function f is defined as

$$\Delta_\lambda^\rho f(x) = \sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}f(x+\lambda\nu).$$

By formula (8) of [**CN2**], we have

$$E(x) = E^0(x) = \lambda^{-\rho}\Delta_\lambda^\rho(A^\rho(x) - S_\rho(x)) + O\left(\lambda\log^{m_F-1}x\right) + O\left(\sum_{x<n\leq x+\rho\lambda}|a_n|\right) \quad (19)$$

Notations 6. — *In case $d > 1$, we define $\lambda(x) = x^{1-\delta-\eta}$, where $\delta = 1/d$ and $\eta > 0$ and choose a convenient η for each case.*

Definition 6. — *We define $W(x) = \Delta_\lambda^\rho(A^\rho(x) - S_\rho(x))$ and $V(x) = E(x) - \lambda^{-\rho}W(x)$.*

We assume $d > 1$. In order to estimate $W(y)$, we shall use proposition 6.1, the asymptotic expansion of I_ρ and some properties of finite differences. This study follows the method used by Chandrasekharan and Narasimhan in [**CN1**].

Notations 7. — *We write for all natural integer n, $\alpha_n = n/Q^2$.*

Since we have $W(y) = \Delta_\lambda^\rho \left(A^\rho(y) - S_\rho(y)\right)$, by proposition 6.1, we have for $\rho > d/2$ and $y > 0$,

$$W(y) = \alpha Q^{-1} \sum_{n=1}^{\infty} \frac{\overline{a_n}}{\alpha_n^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_n y).$$

We estimate now $\Delta_\lambda^\rho I_\rho(y)$.

Assume that $\rho \in \mathbb{N}$, $\lambda > 0$, $y \in \mathbb{R}$ and f has a ρ^{th} derivative on $\mathbb{R}^{+*}$. Then we have

$$\Delta_\lambda^\rho f(y) = \int_y^{y+\lambda} \int_{t_1}^{t_1+\lambda} \cdots \int_{t_{\rho-1}}^{t_{\rho-1}+\lambda} f^{(\rho)}(t_\rho) dt_\rho \cdots dt_2 dt_1.$$

In particular,

$$|\Delta_\lambda^\rho f(y)| \leq \lambda^\rho \sup_{y \leq t \leq y+\rho\lambda} |f^{(\rho)}(t)|.$$

Applying it to I_ρ, we get by proposition 7.4

$$|\Delta_\lambda^\rho I_\rho(y)| \leq \lambda^\rho \sup_{y \leq t \leq y+\rho\lambda} |I_0(t)|.$$

By proposition 7.3, $I_0(x) = O\left(x^{\frac{1}{2}(1-\delta)}\right)$ so if $y \geq \lambda\rho$, we have

$$|\Delta_\lambda^\rho I_\rho(y)| \leq \lambda^\rho O\left(y^{\frac{1}{2}(1-\delta)}\right). \tag{20}$$

On the other hand, we have by proposition 4

$$|\Delta_\lambda^\rho I_\rho(y)| = O\left(|I_\rho(y)|\right) = O\left(y^{\Re\omega}\right) = O\left(y^{(\rho+1/2)(1-\delta)}\right). \tag{21}$$

We have for $\rho > d/2$, $z > 0$ and $y > 0$,

$$W(y) = \alpha Q^{-1} \sum_{n=1}^{\infty} \frac{\overline{a_n}}{\alpha_n^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_n y) \tag{22}$$

$$= \alpha Q^{-1} \sum_{\alpha_n \leq z} \frac{\overline{a_n}}{\alpha_n^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_n y) + \alpha Q^{-1} \sum_{\alpha_n > z} \frac{\overline{a_n}}{\alpha_n^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_n y).$$

Using (20) in the first sum and (21) in the second one, we get

$$W(y) = O\left(\sum_{\alpha_n \leq z} \frac{|a_n|}{\alpha_n^{1+\rho}} \alpha_n^\rho \lambda^\rho (\alpha_n y)^{\frac{1}{2}(1-\delta)}\right) + O\left(\sum_{\alpha_n > z} \frac{|a_n|}{\alpha_n^{1+\rho}} (\alpha_n y)^{(\rho+1/2)(1-\delta)}\right)$$

$$= O\left(\sum_{\alpha_n \leq z} \frac{|a_n|}{\alpha_n^{\frac{1}{2}(1+\delta)}} \lambda^\rho y^{\frac{1}{2}(1-\delta)}\right) + O\left(\sum_{\alpha_n > z} \frac{|a_n|}{\alpha_n^{\rho\delta+\frac{1}{2}(1-\delta)}} y^{(\rho+1/2)(1-\delta)}\right).$$

If $\rho > \frac{d+1}{2}$, using $\lambda = y^{1-\delta-\eta}$ and choosing $z = y^{d\eta}$, since for all $\varepsilon > 0$, $\sum_{n=1}^{+\infty} |a_n| n^{-1-\varepsilon} < +\infty$, we get for all $\varepsilon > 0$

$$W(y) = O\left(z^{1+\varepsilon-\frac{1}{2}(1+\delta)} y^{(\rho+1/2)(1-\delta)-\rho\eta}\right) + O\left(z^{1+\varepsilon-\rho\delta-\delta/2} y^{(\rho+1/2)(1-\delta)}\right)$$

$$W(y) = O\left(y^{(\rho+1/2)(1-\delta-\eta)+d\eta/2+\varepsilon}\right). \tag{23}$$

We can now prove theorem 2. Let $F \in S^{\#}$ such that for all $\varepsilon > 0$, we have $\sum_{n\leq x} |a_n|^2 \ll n^{1+\varepsilon}$.

By Cauchy's inequality, we have

$$\left(\sum_{y<n\leq y+\rho\lambda} |a_n|\right)^2 \leq \rho\lambda \left(\sum_{y<n\leq y+\rho\lambda} |a_n|^2\right) \ll y^{2-\delta-\eta+\varepsilon}, \tag{24}$$

for all $\varepsilon > 0$, so we have

$$V(y) \ll y^{1-\frac{1}{2}(\delta+\eta)+\varepsilon}. \tag{25}$$

Since $E(x) = V(x) + \lambda^{-\rho}W(x)$, we have by (25) and (23), for all $\varepsilon > 0$,

$$E(x) = O\left(x^{(1-\delta-\eta)/2+d\eta/2+\varepsilon}\right) + O\left(x^{1-\frac{1}{2}(\delta+\eta)+\varepsilon}\right).$$

We choose $\eta = \delta$ and we get $E(x) = O\left(x^{1-\delta+\varepsilon}\right)$.

If we assume that for all $\varepsilon > 0$, $a_n = O\left(n^{\varepsilon}\right)$, then $V(x) \ll x^{1-\delta-\eta+\varepsilon}$, so we have by (23), for all $\varepsilon > 0$,

$$E(x) = O\left(x^{(1-\delta-\eta)/2+d\eta/2+\varepsilon}\right) + O\left(x^{1-\frac{1}{2}(\delta+\eta)+\varepsilon}\right).$$

We choose $\eta = \frac{1-\delta}{d+1}$ and we get $E(x) = O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right)$.

9. Estimation of the mean square of the error term

We shall now estimate the integral $\int_0^x |E(y)|^2 dy$ and prove theorem 3. Since the estimate of theorem 3 is obvious for functions of degree $d \leq 1$ in $S^{\#}$, we shall assume that $F \in S^{\#}$ is a function of degree $d > 1$ such that for all *varepsilon* > 0, $\sum_{n\leq x} |a_n|^2 \ll x^{1+\varepsilon}$. We have

$$\int_1^x |E(y)|^2 dy \leq 4 \max\left(\int_1^x \lambda^{-2\rho}|W(y)|^2 dy, \int_1^x |V(y)|^2 dy\right).$$

9.1. Estimation of the second integral.

— When x tends to infinity, we have

$$\int_2^x \lambda(y)^2 \log^{2m_F-2} y dy = \int_2^x y^{2(1-\delta-\eta)} \log^{2m_F-2} y dy \ll x^{3-2\delta-2\eta} \log^{2m_F-2} x + 1.$$

By (24), we have

$$\int_1^x \left(\sum_{y<n\leq y+\rho\lambda} |a_n|\right)^2 dy = O\left(\int_1^x y^{1-\delta-\eta}\left(\sum_{y<n\leq y+\rho\lambda} |a_n|^2\right) dy\right)$$

$$= O\left(\sum_{1<n\leq x+\rho x^{1-\delta-\eta}} |a_n|^2 \int_{n-\rho n^{1-\delta-\eta}}^n y^{1-\delta-\eta} dy\right) = O\left(\sum_{1<n\leq x+\rho x^{1-\delta-\eta}} |a_n|^2 n^{2(1-\delta-\eta)}\right).$$

So, for all $\varepsilon > 0$, we have when x tends to infinity

$$\int_1^x \left(\sum_{y<n\leq y+\rho\lambda} |a_n| \right)^2 dy = O\left((x+\rho x^{1-\delta-\eta})^{2(1-\delta-\eta)+1+\varepsilon} \right) = O(x^{3-2\delta-2\eta+\varepsilon}).$$

Finaly, by (19), for all $\varepsilon > 0$, we have

$$\int_1^x |V(y)|^2 dy = O(x^{3-2\delta-2\eta+\varepsilon}+1). \tag{26}$$

9.2. Estimation for the first integral. — In order to estimate $\int_1^x \lambda^{-2\rho}|W(y)|^2 dy$, we shall use proposition 6.1, the asymptotic expansion of I_ρ and some properties of finite differences. This study follows the method used by Chandrasekharan and Narasimhan in **[CN2]**.

By (22), for $\rho > d/2$ and $y > 0$, we have

$$|W(y)|^2 = Q^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\overline{a_m}a_n}{(\alpha_m\alpha_n)^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_m y)\Delta_\lambda^\rho \overline{I_\rho}(\alpha_n y). \tag{27}$$

Definition 7. — *We write*

$$W_1(y) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\alpha_n^{2(1+\rho)}} \left|\Delta_\lambda^\rho I_\rho\left(\alpha_n y\right)\right|^2,$$

$$W_2(y) = \sum_{n\neq m} \frac{\overline{a_m}a_n}{(\alpha_m\alpha_n)^{1+\rho}} \Delta_\lambda^\rho I_\rho\left(\alpha_m y\right) \Delta_\lambda^\rho \overline{I_\rho}\left(\alpha_n y\right).$$

We shall estimate $\int_1^x \lambda^{-2\rho}W_1(y)dy$ and $\int_1^x \lambda^{-2\rho}W_2(y)dy$ separately.

9.2.1. Estimation of diagonal terms. —

Proposition 9.1. — *If $\rho \in \mathbb{N}$ and $x > 1$, then*

$$\int_1^x \lambda^{-2\rho}W_1(y)dy = O\left(x^{2-\delta}\right).$$

Proof By (20), we have

$$\int_1^x \lambda^{-2\rho}W_1(y)dy = \int_1^x \lambda^{-2\rho}\sum_{n\geq 1}\frac{|a_n|^2\left|\Delta_\lambda^\rho I_\rho\left(\alpha_n y\right)\right|^2}{\alpha_n^{2(1+\rho)}}dy$$

$$= \int_1^x \lambda^{-2\rho}\sum_{n\geq 1}\frac{|a_n|^2\left(\lambda^\rho\alpha_n^\rho\left(\alpha_n y\right)^{\frac{1}{2}(1-\delta)}\right)^2}{\alpha_n^{2(1+\rho)}}dy \ll \int_1^x \sum_{n\geq 1}\frac{|a_n|^2}{\alpha_n^{1+\delta}}y^{1-\delta}dy.$$

For all $\varepsilon > 0$, $\sum_{\alpha_n\leq x}|a_n|^2 = O_\varepsilon\left(x^{1+\varepsilon}\right)$, so if $\varepsilon < \delta$, $\sum_{n\geq 1}\frac{|a_n|^2}{\alpha_n^{1+\delta}} = O_\delta(1)$. This proves

$$\int_1^x \sum_{n\geq 1}\frac{|a_n|^2}{\alpha_n^{1+\delta}}y^{1-\delta}dy = O_\delta\left(x^{2-\delta}\right). \qquad \square$$

9.2.2. Estimation of non-diagonal terms. —

Notations 8. — *We recall that* $\omega = (\rho + \frac{1}{2})(1-\delta) - i\delta\Im\xi$, $\alpha_n = n/Q^2$ *and* $a = d\beta^\delta$.

The equation (27) shows that the estimation of the non-diagonal terms reduces to estimating the integral

$$\int_1^x \lambda(y)^{-2\rho} \Delta_\lambda^\rho I_\rho\left(\alpha_m y\right) \Delta_\lambda^\rho \overline{I_\rho}\left(\alpha_n y\right) dy.$$

Using the asymptotic expansion of I_ρ given in proposition 4, this estimate is reduced to that of the integral

$$U_{m,n}^{(b,c)}(x) = \int_1^x \lambda(y)^{-2\rho} \Delta_\lambda^\rho \left(y^\omega e^{iby^\delta \alpha_m^\delta}\right) \Delta_\lambda^\rho \left(y^{\overline{\omega}} e^{-icy^\delta \alpha_n^\delta}\right) dy, \tag{28}$$

where b and c are two real numbers such that $|b| = |c| = a$.

Lemma 8. — *For* $\delta < 1$, $\rho \geq 1$, $z > 1$ *and* $x > \max\left(1, (2\rho)^{\frac{1}{\eta+\delta}}\right)$, *we have*

$$\begin{aligned}
U_{m,n}^{b,c}(x) &= O\left(\frac{1}{\alpha_m^\delta - \alpha_n^\delta}\right) x^{2(1-\delta)} (\alpha_m \alpha_n)^{\delta\rho} && \text{if } \alpha_n < \alpha_m < z && (29)\\
&= O\left(\frac{\alpha_n^{\delta\rho}}{\alpha_m^\delta - \alpha_n^\delta}\right) x^{(\rho+2)(1-\delta)} \left(1 + \lambda x^{\delta-1}\alpha_m^\delta\right) && \text{if } \alpha_n \leq z < \alpha_m && (30)\\
&= O\left(\frac{1}{\alpha_m^\delta - \alpha_n^\delta}\right) \lambda^{-2\rho} x^{(2\rho+1)(1-\delta)} \left(x^{1-\delta} + \lambda\alpha_m^\delta\right) && \text{if } z < \alpha_n < \alpha_m. && (31)
\end{aligned}$$

Proof In this proof, m and n are fixed so we do not always write them. For $b, c \in \mathbb{R}^*$ such that $|b| = |c|$ and for $y > 0$, we define

$$G_m^b(y) = e^{-iby^\delta \alpha_m^\delta} \Delta_\lambda^\rho \left(y^\omega e^{iby^\delta \alpha_m^\delta}\right) \text{ and } H^{b,c}(y) = \lambda(y)^{-2\rho} y^{1-\delta} G_m^b(y) \overline{G_n^c(y)},$$

so we have

$$\begin{aligned}
U_{m,n}^{b,c}(x) &= \int_1^x \lambda(y)^{-2\rho} G_m^b(y) \overline{G_n^c(y)} e^{iy^\delta (b\alpha_m^\delta - c\alpha_n^\delta)} dy\\
&= \frac{1}{i\delta(b\alpha_m^\delta - c\alpha_n^\delta)} \int_1^x H^{b,c}(y) \frac{d}{dy}\left(e^{iy^\delta(b\alpha_m^\delta - c\alpha_n^\delta)}\right) dy,
\end{aligned}$$

so

$$U_{m,n}^{b,c}(x) = O\left(\frac{1}{|\alpha_m^\delta - \alpha_n^\delta|}\right)\left(|H^{b,c}(x)| + |H^{b,c}(1)| + \int_1^x |\frac{d}{dy} H^{b,c}(y)| dy\right). \tag{32}$$

We have

$$|G_m^b(y)| = \left|\Delta_\lambda^r \left(e^{iby^\delta \alpha_m^\delta} y^\omega\right)\right| \text{ and } \left|\frac{d}{dy} G_m^b(y)\right| = \left| e^{ib\alpha_m^\delta y^\delta} \frac{d}{dy}\left(e^{-ib\alpha_m^\delta y^\delta} \Delta_\lambda^\rho\left(y^\omega e^{ib\alpha_m^\delta y^\delta}\right)\right)\right|$$

and these two functions have been estimated par K. Chandrasekharan and R. Narasimhan in **[CN2]** in case ω is real. In case $\omega \in \mathbb{C}$, the estimations and their proof are

the same (see [**DR**] for more details). We recall here their results :
If $y > \max\left(1, (2\rho)^{\frac{1}{\eta+\delta}}\right)$, we have

$$G_m^b(y) = O\left(y^{\Re\omega}\right) \tag{33}$$

$$= y^\omega \Delta_\lambda^\rho\left(e^{ib\alpha_m^\delta y^\delta}\right) + O\left(\lambda y^{\Re\omega-1}\right) \tag{34}$$

$$= O\left(\lambda^\rho y^{\Re\omega-\rho(1-\delta)}\alpha_m^{\delta\rho}\right). \tag{35}$$

$$\frac{d}{dy}G_m^b(y) = O\left(\lambda\alpha_m^\delta y^{\delta-2+\Re\omega}\right) \qquad \text{pour } \rho \geq 1 \tag{36}$$

$$= O\left(\lambda^\rho y^{\Re\omega-\rho(1-\delta)-1}\alpha_m^{\delta\rho}\right). \tag{37}$$

Using $\lambda'(y) = O\left(\frac{\lambda(y)}{y}\right)$, we get

$$\left|\frac{d}{dy}H^{b,c}(y)\right| = O\left(y^{-1}H^{b,c}(y)\right)$$

$$+ \lambda(y)^{-2\rho}y^{1-\delta}O\left(\left|\frac{d}{dy}G_m^b(y)\right||G_n^c(y)| + |G_m^b(y)|\left|\frac{d}{dy}G_n^c(y)\right|\right). \tag{38}$$

– If $\alpha_n < \alpha_m \leq z$, by (35) and (37) we have

$$H_{k,l}^{b,c}(y) = \lambda^{-2\rho}y^{1-\delta}O\left(\lambda^{2\rho}y^{2\Re\omega-2\rho(1-\delta)}(\alpha_m\alpha_n)^{\delta\rho}\right) = O\left(y^{2(1-\delta)}(\alpha_m\alpha_n)^{\delta\rho}\right) \tag{39}$$

and by (38)

$$\frac{d}{dy}H^{b,c}(y) = O\left(\lambda^{-2\rho}y^{-\delta}\lambda^{2\rho}y^{2\Re\omega-2\rho(1-\delta)}(\alpha_m\alpha_n)^{\delta\rho}\right) = O\left(y^{1-2\delta}(\alpha_m\alpha_n)^{\delta\rho}\right). \tag{40}$$

By (39), (40) and (32), we get (29).

– If $\alpha_n \leq z < \alpha_m$, by (33) applied to $G_m^b(y)$, (35) to $G_n^c(y)$, (36) to $\frac{d}{dy}G_m^{k,b}(y)$ and (37) to $\frac{d}{dy}G_n^{l,c}(y)$, we have

$$H^{b,c}(y) = \lambda^{-2\rho}y^{1-\delta}O\left(\lambda^\rho y^{2\Re\omega-\rho(1-\delta)}\alpha_n^{\delta\rho}\right) = O\left(\lambda^{-\rho}y^{(\rho+2)(1-\delta)}\alpha_n^{\delta\rho}\right) \tag{41}$$

and by (38)

$$\frac{d}{dy}H^{b,c}(y) = O\left(\lambda^{-\rho}y^{2\Re\omega-(\rho-1)(1-\delta)-1}\alpha_n^{\delta\rho}\right)$$

$$+\lambda^{-2\rho}y^{1-\delta}O\left(\lambda y^{\Re\omega-2+\delta}\alpha_m^\delta\lambda^\rho y^{\Re\omega-\rho(1-\delta)}\alpha_n^{\delta\rho} + y^{\Re\omega}\lambda^\rho y^{\Re\omega-1-\rho(1-\delta)}\alpha_n^{\rho\delta}\right)$$

$$= O\left(\lambda^{-\rho}y^{-\delta+2\Re\omega-\rho(1-\delta)}\alpha_n^{\delta\rho}\right) + O\left(\lambda^{-\rho+1}y^{2\Re\omega-\rho(1-\delta)-1}\alpha_m^\delta\alpha_n^{\delta\rho}\right)$$

$$= O\left(\lambda^{-\rho}y^{(\rho+1)(1-\delta)-\delta}\alpha_n^{\delta\rho}\left(1+\lambda y^{\delta-1}\alpha_m^\delta\right)\right). \tag{42}$$

By (41), (42) and (32), we get (30).

– If $z < \alpha_n < \alpha_m$, by (33) and (36), we have

$$|H^{b,c}(y)| = \lambda^{-2\rho} y^{1-\delta} O\left(y^{2\Re\omega}\right) = O\left(\lambda^{-2\rho} y^{(2\rho+2)(1-\delta)}\right). \tag{43}$$

and by (38)

$$\frac{d}{dy} H^{b,c}(y) = O\left(\lambda^{-2\rho} y^{2\Re\omega-\delta}\right) + \lambda^{-2\rho+1} y^{2\Re\omega-1} O\left(\alpha_m^\delta + \alpha_n^\delta\right)$$

so

$$\frac{d}{dy} H^{b,c}(y) = O\left(\lambda^{-2\rho} y^{(2\rho+1)(1-\delta)-\delta}\left(1 + \lambda y^{\delta-1}\alpha_n^\delta\right)\right). \tag{44}$$

By (43), (44) and (32), we have (31). □

We can deduce from these estimations those of non-diagonal terms. The proof is the same as that of Chandrasekharan and Narasimhan in [**CN2**], we just recall here their conclusions.

Lemma 9. — *Assume that*

$$W_2(x) = \sum_{m \neq n} \frac{b_m \overline{b_n}}{(\alpha_m \alpha_n)^{1+\rho}} \Delta_\lambda^\rho I_\rho(\alpha_m y) \Delta_\lambda^\rho \overline{I_\rho}(\alpha_n y),$$

that I_ρ expands asymtotically to the form

$$I_\rho(x) \sim e^{iax^\delta} \sum_{\nu=0}^{M} \delta_\nu x^{\omega-\nu\delta} + e^{-iax^\delta} \sum_{\nu=0}^{M} \delta'_\nu x^{\omega-\nu\delta},$$

and that the estimations of the lemma 8 hold for the function

$$U_{m,n}^{(b,c)}(x) = \int_1^x \lambda(y)^{-2\rho} \Delta_\lambda^\rho\left(y^\omega e^{iby^\delta \alpha_m^\delta}\right) \Delta_\lambda^\rho\left(y^{\overline{\omega}} e^{-icy^\delta \alpha_n^\delta}\right) dy,$$

for all $b, c \in \mathbb{R}$ such that $|b| = |c| = a$.
Then for all $\varepsilon > 0$, we have

$$\int_1^x \lambda^{-2\rho} W_2(y) dy = O\left(x^{2(1-\delta)}\left(x^{\eta(d-2)+\varepsilon} + \log x\right)\right).$$

9.3. Proof of theorem 3. — With the hypothesis of theorem 3, we have

$$\int_1^x \lambda^{-2\rho} |W(y)|^2 dy = O\left(x^{2-\delta}\right) + O\left(x^{2(1-\delta)}\left(x^{\eta(d-2)+\varepsilon} + \log x\right)\right)$$

and

$$\int_1^x |V(y)|^2 dy = O(x^{3-2\delta-2\eta+\varepsilon}).$$

Finally, for all $\eta > 0$, we have

$$\int_1^x |E(y)|^2 dy = O\left(x^{2-\delta}\right) + O\left(x^{2(1-\delta)}\left(x^{\eta(d-2)+\varepsilon} + \log x\right)\right) + O(x^{3-2\delta-2\eta+\varepsilon}).$$

– If $d < 2$, then we choose η tending to infinity and we have

$$\int_1^x |E(y)|^2 dy = O\left(x^{2-\delta}\right).$$

– If $d \geq 2$, then we choose $\eta = \delta$ and we have

$$\int_1^x |E(y)|^2 dy = O\left(x^{2-\delta}\right) + O(x^{3-4\delta+\varepsilon}).$$

This ends the proof.

Estimates of theorem 3 prove that for $d < 4$, the function $E(x)$ belongs to $L^2([0, +\infty[, dx/x^2)$. We shall use it in [**DR1**]. The estimation we get in this last theorem is similar to the one obtained by Chandrasekharan and Narasimhan in [**CN2**]. We intend to adapt the method used by Redmond in [**R2**] in order to sharpen our result in the case $d \geq 4$ and avoid the use of the hypothesis $\sum_{n \leq x} |a_n|^2 \ll x^{1+\varepsilon}$..

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